Problem set 3

Due date: 14th Sep

Exercise 12. [*Radon Nikodym theorem:*] Let $(\Omega, \mathcal{F})$ be a measure space and let μ, ν be two sigma-finite measures on the σ-algebra $\mathcal{F}$. If $\nu(A) = 0$ implies $\mu(A) = 0$, we say that $\mu \ll \nu$. Radon-Nikodym theorem asserts that if $\mu \ll \nu$ then there is a non-negative measurable function g such that $\mu(A) = \int_A g d\nu$ for all $A \in \mathcal{F}$.

Prove this as follows. Assume throughout that μ and ν are probability measures (i.e., $\mu(\Omega) = \nu(\Omega) = 1$).

(1) Let $\theta = \mu + \nu$. Show that $\exists h \in L^2(\theta)$ such that $\int f \, d\mu = \int fh \, d\theta$ for every $f \in L^2(\theta)$.

(2) Show that $0 \le h \le 1$ *a.e[μ]*. [*Hint:* Consider $\mathbf{1}_{h<0}$ and $\mathbf{1}_{h>1}$].

(3) Use the first two parts to show that $\int f(1-h) \, d\mu = \int fh \, d\nu$ for any measurable $f \ge 0$.

(4) Set $g := \frac{h}{1-h}$ to get the conclusion.

Did you use the absolute continuity of μ w.r.t ν? If not go back to step 2.

[**Note:** Obviously the same holds if μ and ν are any finite measures (then $\mu(\Omega)^{-1}\mu$ and $\nu(\Omega)^{-1}\nu$ are probability measures). A little more work extends it to σ-finite measures $\mu \ll \nu$].

Exercise 13. For $1 \le p < \infty$, show that $(\ell^p)^* \cong \ell^q$ where $\frac{1}{p} + \frac{1}{q} = 1$. Follow these steps.

(1) If $\alpha \in \ell^q$, show that $L_\alpha(x) = \sum \alpha_k x_k$ defines a bounded linear functional on ℓ^p with $\|L_\alpha\| = \|\alpha\|_q$.

(2) If $L \in (\ell^p)^*$, find $\alpha = (\alpha_1, \alpha_2, \ldots)$ such that $Lx = \sum \alpha_k x_k$ for $x \in \ell^p$. (Make suitable noises before writing infinite sums).

(3) Show that the α from part (2) is in ℓ^q and that $\|\alpha\|_q = \|L\|$.

Exercise 14. Consider ℓ^∞.

(1) For any $\alpha \in \ell^1$, show that $L_\alpha x = \sum \alpha_k x_k$ defines a bounded linear functional on ℓ^∞ with $\|L_\alpha\| = \|\alpha\|_1$.

(2) Using Hahn-Banach theorem or otherwise, show that there exists $L \in (\ell^\infty)^*$ that is not of the form L_α for any $\alpha \in \ell^1$.